Ch. 10.1

Vector (Valued) Functions

A vector-valued function $\vec{F}$ of a real variable with domain D assigns to each number t in the set D a unique vector $\vec{F}(t)$

The set of all vectors $\vec{v}$ of the form:

$$\vec{v} = \vec{F}(t)$$

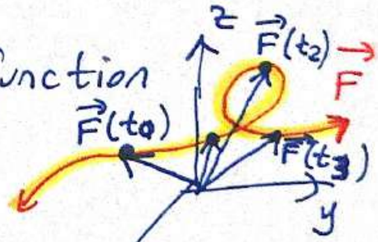
for t in D $(t \in D)$ in the range of $\vec{F}$. That is

$$\vec{F}(t) = f_1(t)\vec{i} + f_2(t)\vec{j} + f_3(t)\vec{k} \quad (\text{in } \mathbb{R}^3)$$

Here, f_i are real-valued (scalar) functions of $(t \in \mathbb{R})$ the real number t defined on the domain set D.

A vector function can also be described by

$$\vec{F}(t) = \langle f_1(t), f_2(t), f_3(t) \rangle.$$



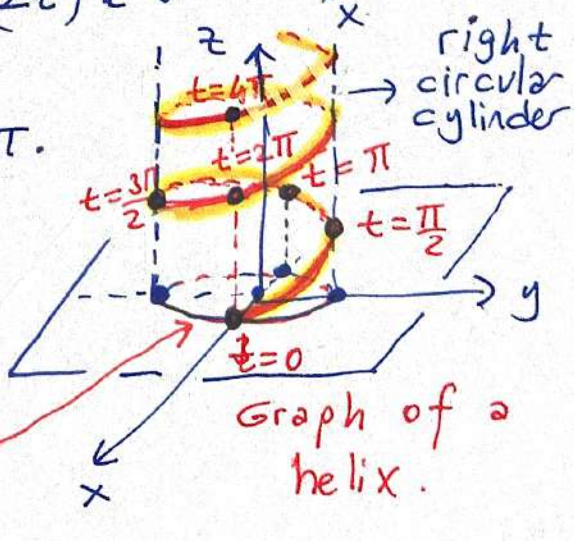
Ex. Sketch the graph of the vector function

$(t \in [0, 4\pi])$ $\vec{F}(t) = \langle 3\cos t, 3\sin t, 2t \rangle$.

$$\vec{F}(t) = (3\cos t)\vec{i} + (3\sin t)\vec{j} + (2t)\vec{k}.$$

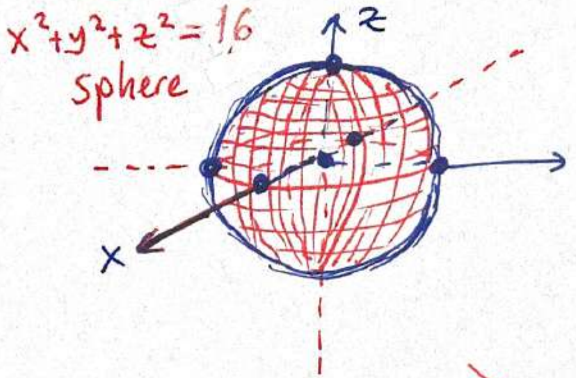
$$\left.\begin{aligned} x(t) &= 3\cos t \\ y(t) &= 3\sin t \\ z(t) &= 2t \end{aligned}\right\} \text{for all } 0 \le t \le 4\pi.$$

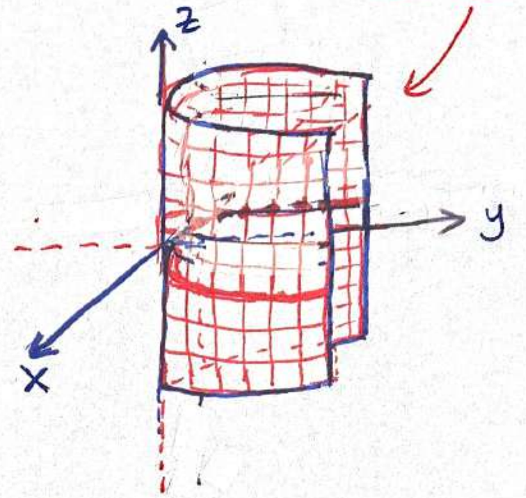
$$x^2 + y^2 = 9\cos^2 t + 9\sin^2 t$$
$$= 9(\cos^2 t + \sin^2 t)$$
$$x^2 + y^2 = 9$$



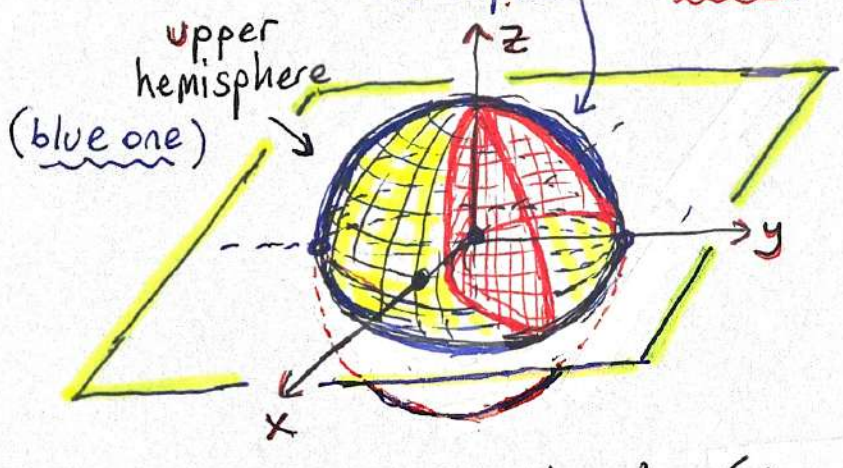
Graph of a helix.

Ex : Find a vector-valued function $\vec{F}$ whose graph is the curve of the intersection of the hemisphere $z=\sqrt{16-x^2-y^2}$ and the parabolic cylinder $y=x^2$





a part of parabolic cylinder inside the (upper) hemisphere (red one)



intersection curve of the upper hemisphere and parabolic cylinder on sphere

When $x(t)=t$, then $y(t)=t^2$. (from $y=x^2$)

Substituting it into the hemisphere $\left(y(t)=t^2=x^2(t)\right)$

(then)

$$\Rightarrow \quad z=\sqrt{16-x^2-y^2}$$

$$\Rightarrow \quad z(t)=\sqrt{16-(t)^2-(t^2)^2}$$

$$\Rightarrow \quad z(t)=\sqrt{16-t^2-t^4}.$$

$$16-t^2-t^4 \geqslant 0$$

So, $\vec{F}(t)=\langle x(t), y(t), z(t)\rangle=\langle t, t^2, \sqrt{16-t^2-t^4}\rangle$

Vector Function Operations

Let $\vec{F}$ and $\vec{G}$ be vector functions of the real variable t, and let $f(t)$ be a scalar function. Therefore, $\vec{F}+\vec{G}$, $\vec{F}-\vec{G}$, $f.\vec{F}$ and $\vec{F}\times\vec{G}$ are vector functions and $\vec{F}\bullet\vec{G}$ is a scalar function, and then ;

- $(\vec{F}+\vec{G})(t) = \vec{F}(t)+\vec{G}(t)$,
- $(\vec{F}-\vec{G})(t) = \vec{F}(t)-\vec{G}(t)$,
- $(f.\vec{F})(t) = f(t).\vec{F}(t)$,
- $(\vec{F}\times\vec{G})(t) = \vec{F}(t)\times\vec{G}(t)$,
- $(\vec{F}\bullet\vec{G})(t) = \vec{F}(t)\bullet\vec{G}(t)$ (scalar function).

All this operations are determined on the intersection of the domains of the functions that compose the operation.

Ex : Find the following using

$$\vec{F}(t) = t^3\vec{i} - (\cos t)\vec{j} + t\,\vec{k},$$
$$\vec{G}(t) = -\frac{1}{2}t\,\vec{i} + t^2\vec{j} + \sin t\,\vec{k}.$$

- $(\vec{F}+\vec{G})(t) = (t^3-\frac{1}{2}t)\vec{i} + (-\cos t + t^2)\vec{j} + (t+\sin t)\vec{k}$
- $(F-G)(t) = (t^3+\frac{1}{2}t)\vec{i} + (-\cos t - t^2)\vec{j} + (t-\sin t)\vec{k}$
- $(F\times G)(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ t^3 & -\cos t & t \\ -\frac{1}{2}t & t^2 & \sin t \end{vmatrix}$

$$= \left[(-\cos t)(\sin t) - t^2.t\right]\vec{i} - \left[t^3.\sin t - \frac{1}{2}t.t\right]\vec{j} + \left[t^3.t^2 - \left(-\frac{1}{2}t\right)(-\cos t)\right]\vec{k}$$

$$= \left(-\cos t\sin t - t^3\right)\vec{i} + \left(\frac{t^2}{2} - t^3\sin t\right)\vec{j} + \left(t^5 - \frac{t}{2}\cos t\right)\vec{k}$$

- $(\vec{F}\bullet\vec{G})(t) = \vec{F}(t)\bullet\vec{G}(t)$

$$= t^3.\left(-\frac{1}{2}t\right) + (-\cos t)t^2 + t\sin t.$$

Limit on Vector Functions:

$$\lim_{t \to t_0} \vec{F}(t) = \lim_{t \to t_0} \langle f_1(t), f_2(t), f_3(t) \rangle$$

$$\Rightarrow \boxed{\lim_{t \to t_0} \vec{F}(t)} = \boxed{\langle \lim_{t \to t_0} f_1(t), \lim_{t \to t_0} f_2(t), \lim_{t \to t_0} f_3(t) \rangle}$$

(The limit of $\vec{F}(t)$ exists) (iff) $\Longleftrightarrow$ (The limit of each component exists)

EX. $\vec{F}(t) = \langle \underbrace{2+t^3}_{f_1(t)}, \underbrace{t^2 . e^{-t}}_{f_2(t)}, \underbrace{\frac{\sin t}{t}}_{f_3(t)} \rangle$ then find the $\lim_{t \to 0} \vec{F}(t)$.

$f_1(t) = 2 + t^3$,

$f_2(t) = t^2 . e^{-t}$,

$f_3(t) = \frac{\sin t}{t}$.

$$\lim_{t \to 0} f_1(t) = \lim_{t \to 0} (2+t^3) = 2 + (0)^3 = 2,$$

$$\lim_{t \to 0} f_2(t) = \lim_{t \to 0} (t^2 . e^{-t}) = (0)^2 . e^0 = 0 . 1 = 0,$$

$$\lim_{t \to 0} f_3(t) = \lim_{t \to 0} \left(\frac{\sin t}{t}\right) = 1.$$

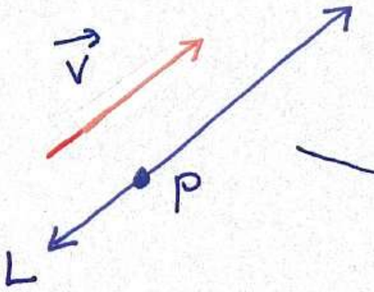
$$\Rightarrow \lim_{t \to 0} \vec{F}(t) = \langle 2, 0, 1 \rangle$$

$$= 2\vec{i} + \vec{k}.$$

Ex. Find the curve using the vector function

$$\vec{r}(t) = \langle 3+t, 2-3t, 1+7t \rangle .$$

$$\left.\begin{aligned} x(t) &= 3+t \\ y(t) &= 2-3t \\ z(t) &= 1+7t \end{aligned}\right\}$$ then, it is the line equation passing through point $P(3,2,1)$ and parallel to $\vec{v} = \langle 1, -3, 7 \rangle$.



$\rightarrow$ { a line is a special case of a curve. }

Ex. Find the following limit

$$\lim_{t\to 1} \left\langle \frac{t^3-1}{t-1}, \frac{t^2-3t+2}{t^2+t-2}, (t^2+1)e^{t-1} \right\rangle .$$

- $\displaystyle \lim_{t\to 1} \frac{t^3-1}{t-1} = \lim_{t\to 1} \frac{\cancel{(t-1)}(t^2+t+1)}{\cancel{t-1}} = (1)^2+(1)+1 = 3,$
- $\displaystyle \lim_{t\to 1} \frac{t^2-3t+1}{t^2+t-2} = \lim_{t\to 1} \frac{\cancel{(t-1)}(t-2)}{\cancel{(t-1)}(t+2)} = \frac{1-2}{1+2} = -\frac{1}{3},$
- $\displaystyle \lim_{t\to 1} (t^2+1)e^{t-1} = ((1)^2+1)\,e^{1-1} = 2 . e^0 = 2.1 = 2 .$

$$\Rightarrow \lim_{t\to 1} \vec{F}(t) = \left\langle 3, -\frac{1}{3}, 2 \right\rangle \quad \text{or}$$

$$= 3\vec{i} - \frac{1}{3}\vec{j} + 2\vec{k} .$$

Ex. Find the limit

$$\lim_{t\to 0}\vec{F}(t) = \lim_{t\to 0}\left\langle \underbrace{\frac{1-\cos t}{t^2}}_{f_1(t)}, \underbrace{\frac{\sin t\cos t}{2t}}_{f_2(t)}, \underbrace{t.\ln t}_{f_3(t)} \right\rangle.$$

for $f_1(t)$

$$\lim_{t\to 0}\frac{1-\cos t}{t^2} = \frac{1-\cos(0)}{(0)^2} = \frac{1-1}{0} = \frac{0}{0} \text{ undeterminate}$$

Using L'Hospital's rule :

$$\lim_{t\to 0}\frac{0-(-\sin t)}{2t} = \lim_{t\to 0}\frac{\sin t}{2t} = \frac{1}{2}.\underbrace{\lim_{t\to 0}\frac{\sin t}{t}}_{=1} = \frac{1}{2},$$

for $f_2(t)$

$$\lim_{t\to 0}\frac{\sin t.\cos t}{2t} = \frac{\sin(0).\cos(0)}{2.(0)} = \frac{0.1}{0} = \frac{0}{0} \text{ undeterminate}$$

Again L'Hospital :

$$\lim_{t\to 0}\frac{\cos t.\cos t+(-\sin t)\sin t}{2} = \lim_{t\to 0}\frac{\cos^2 t-\sin^2 t}{2}$$

$$= \lim_{t\to 0}\frac{\cos^2(0)-\sin^2(0)}{2} = \frac{1-0}{2} = \frac{1}{2},$$

or

$$\lim_{t\to 0}\frac{2.\sin t.\cos t}{2.2t} = \lim_{t\to 0}\frac{\sin(2t)}{2.2t} = \frac{1}{2}\lim_{\substack{t\to 0\\(2t\to 0)}}\frac{\sin(2t)}{2t}$$

$$= \frac{1}{2}.\underbrace{\lim_{2t\to 0}\frac{\sin(2t)}{2t}}_{=1} = \frac{1}{2},$$

Remember!

$$\boxed{\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1}$$

for $f_3(t)$

$$\lim_{t\to 0} t.\ln t = 0.\infty$$

$$= \lim_{t\to 0}\frac{\ln t}{\frac{1}{t}} = \frac{\infty}{\infty}$$

L'Hospital :

$$\lim_{t\to 0}\frac{1/t}{-\frac{1}{t^2}} = \lim_{t\to 0}\frac{1}{t}.(-t^2) = \lim_{t\to 0}(-t) = 0.$$

So, $\lim\limits_{t \to 0} \vec{F}(t) = \langle \frac{1}{2}, \frac{1}{2}, 0 \rangle$

$$= \frac{1}{2}\vec{i} + \frac{1}{2}\vec{j}$$

Ex. Find the domain of $\vec{F}(t) = \underbrace{\frac{1}{t^2-1}}_{f_1(t)}\vec{i} + \underbrace{\tan t}_{f_2(t)}\vec{j} + \underbrace{\ln t}_{f_3(t)}\vec{k}$.

$D = D_{f_1} \cap D_{f_2} \cap D_{f_3}$

D_{f_1} : $\frac{1}{t^2-1}$ is defined for $t \neq \mp 1$,

D_{f_2} : $\tan t$ is defined for $t \neq \frac{\pi}{2} \mp n\pi$, $n \in \mathbb{N}$

D_{f_3} : $\ln t$ is defined for $t > 0$.

Then, $D = \{t > 0\} \underset{\text{difference}}{\setminus} \{t = 1 \cup t = \frac{\pi}{2} \mp n\pi, n \in \mathbb{N}\}$

Rules for Vector Limits

limit of a

$$\lim_{t\to t_0}\left[\vec{F}(t)+\vec{G}(t)\right] = \lim_{t\to t_0}\vec{F}(t) + \lim_{t\to t_0}\vec{G}(t)\,, \quad \text{(sum)}$$

$$\lim_{t\to t_0}\left[\vec{F}(t)-\vec{G}(t)\right] = \lim_{t\to t_0}\vec{F}(t) - \lim_{t\to t_0}\vec{G}(t)\,, \quad \text{(difference)}$$

$$\lim_{t\to t_0}\left[f(t)\cdot\vec{F}(t)\right] = \lim_{t\to t_0}f(t)\cdot\lim_{t\to t_0}\vec{F}(t)\,, \quad \text{(scalar product)}$$

$$\lim_{t\to t_0}\left[\vec{F}(t)\bullet\vec{G}(t)\right] = \lim_{t\to t_0}\vec{F}(t)\bullet\lim_{t\to t_0}\vec{G}(t)\,, \quad \text{(dot product)}$$

$$\lim_{t\to t_0}\left[\vec{F}(t)\times\vec{G}(t)\right] = \lim_{t\to t_0}\vec{F}(t)\times\lim_{t\to t_0}\vec{G}(t)\,. \quad \text{(cross product)}$$

Continuity for Vector Functions

$\vec{F}(t) = \langle f_1(t), f_2(t), f_3(t)\rangle$ is continuous at t_0 when t_0 is in the domain of the component functions

and

$$\lim_{t\to t_0} f_1(t) = f_1(t_0)\,,$$

$$\lim_{t\to t_0} f_2(t) = f_2(t_0)\,,$$

$$\lim_{t\to t_0} f_3(t) = f_3(t_0)\,.$$

Ex: Is $\vec{F}(t) = \langle \cos t, \ln t, (2-t)^{-1}\rangle$ continuous on natural numbers $\mathbb{N}$?

$$\left.\begin{array}{l} f_1(t) \text{ is continuous for all } t\,, \\ f_2(t) \text{ is continuous for } t>0\,, \\ f_3(t) \text{ is continuous for } t\neq 2\,. \end{array}\right\} \quad \text{So, } \vec{F}(t) \text{ is continuous for } t>0,\ t\neq 2.$$

Ch. 10-2 Derivative of a Vector Function:

The derivative of a vector function $\vec{F}$ is described by

$$\boxed{\lim_{\Delta t \to 0} \frac{\Delta \vec{F}}{\Delta t} = \lim_{\Delta t \to 0} \frac{F(t+\Delta t) - F(t)}{\Delta t}}$$

wherever that limit exists. Here, the derivative vector is represented by $\vec{F}'(t)$ or $\frac{d\vec{F}}{dt}$.

It is called that a vector function $\vec{F}$ is differentiable at $t = t_0$ if $\vec{F}'(t)$ is determined at t_0.
(or defined)

$\left\{ \begin{array}{l} \text{A vector function} \\ \text{is differentiable} \\ \text{called} \end{array} \right\}$ wherever $\left\{ \begin{array}{l} \text{the component functions} \\ f_1, f_2, f_3 \text{ are differentiable} \end{array} \right\}$

(then) $\Rightarrow \vec{F}'(t) = \langle f_1'(t), f_2'(t), f_3'(t) \rangle$. $\left\{ \vec{F}'(t) = \frac{d\vec{F}}{dt} \right\}$

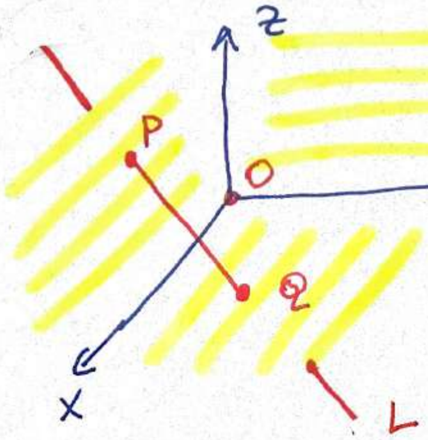
Ex. Is $\vec{H}(t) = \langle \sin t, |t|, t^2 + t + 1 \rangle$ differentiable or not?

$\sin t$ and polynomial functions are diff. for all t.

But $|t|$ is not differentiable for $t = 0$.

Therefore, vector function $\vec{H}(t)$ is not differentiable for all t. But it is differentiable for all $t \neq 0$.

Ex. Find the equation of line which intersects at point $P(3,0,1)$ of xz-plane, $Q(-2,2,0)$ of xy-plane.



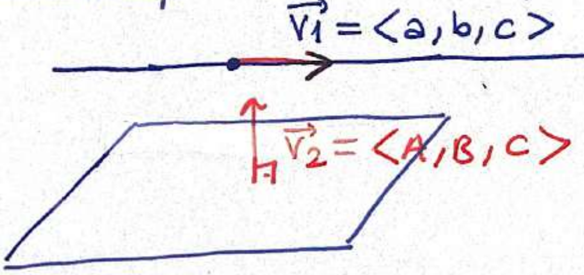
$$\vec{PQ} = (-2-3)\vec{i} + (2-0)\vec{j} + (0-1)\vec{k}$$
$$= -5\vec{i} + 2\vec{j} - \vec{k}$$
$$= \langle \underset{a}{-5}, \underset{b}{2}, \underset{c}{-1} \rangle$$

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}.$$

for P; $\dfrac{x-3}{-5} = \dfrac{y}{2} = \dfrac{z-1}{-1}$ or

for Q; $\dfrac{x-2}{-5} = \dfrac{y-2}{2} = \dfrac{z}{-1}$.

A line parallel to a plane:



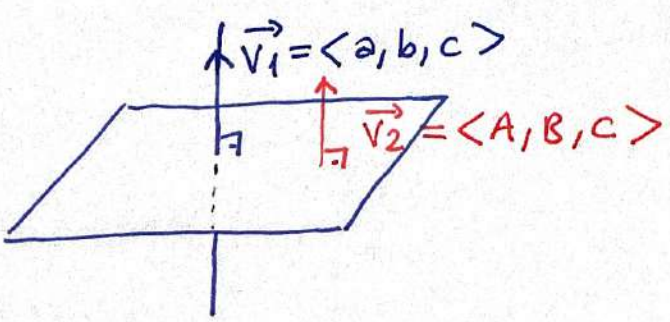
$Ax + By + Cz + D = 0$
equation of a plane

$\vec{V_1} \perp \vec{V_2} \Rightarrow \vec{V_1} \cdot \vec{V_2} = 0$

$\Rightarrow \boxed{aA + bB + cC = 0.}$

parallelity condition for a line and a plane

A line orthogonal to a plane:



$\vec{V_1} \uparrow\uparrow \vec{V_2} \Rightarrow \vec{V_1} = m.\vec{V_2}$

(m: scalar)

$\langle a,b,c \rangle = m \langle A,B,C \rangle$

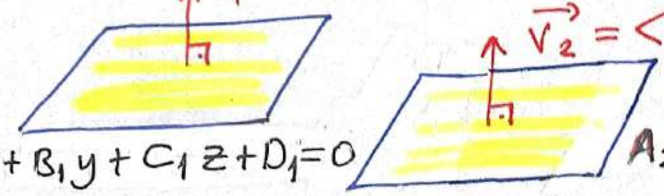
$a = mA$
$b = mB$
$c = mC$

components must be proportional

Parallel Planes

$\vec{V_1} = \langle A_1, B_1, C_1 \rangle$

$\vec{V_2} = \langle A_2, B_2, C_2 \rangle$



$A_1x + B_1y + C_1z + D_1 = 0$ $\quad A_2x + B_2y + C_2z + D_2 = 0$

$\vec{V_1} \uparrow\uparrow \vec{V_2} \implies \vec{V_1} = m.\vec{V_2}$

$\implies \langle A_1, B_1, C_1 \rangle = m.\langle A_2, B_2, C_2 \rangle$

$\implies \boxed{\begin{aligned} A_1 &= m.A_2 \\ B_1 &= m.B_2 \\ C_1 &= m.C_2 \end{aligned}}$. $\left\{ \text{Parallelity condition of two planes.} \right\}$

Orthogonal Planes

$\vec{V_1} \perp \vec{V_2} \implies \vec{V_1} \cdot \vec{V_2} = 0$

$\implies \langle A_1, B_1, C_1 \rangle \cdot \langle A_2, B_2, C_2 \rangle = 0$

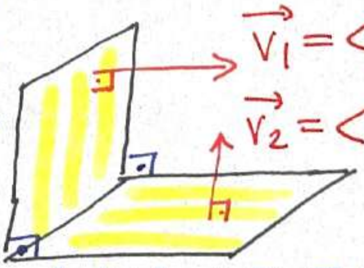
$\implies \boxed{A_1A_2 + B_1B_2 + C_1C_2 = 0}$

$\left\{ \text{orthogonality condition of two planes.} \right\}$

$A_1x + B_1y + C_1z + D_1 = 0$

$\vec{V_1} = \langle A_1, B_1, C_1 \rangle$

$\vec{V_2} = \langle A_2, B_2, C_2 \rangle$

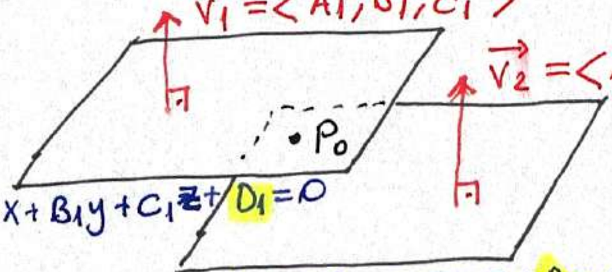


$A_2x + B_2y + C_2z + D_2 = 0$

Coincident Planes

$\vec{V_1} = \langle A_1, B_1, C_1 \rangle$

$\vec{V_2} = \langle A_2, B_2, C_2 \rangle$



$A_1x + B_1y + C_1z + D_1 = 0$

$A_2x + B_2y + C_2z + D_2 = 0$

$\vec{V_1} \uparrow\uparrow \vec{V_2} \implies \vec{V_1} = m.\vec{V_2}$

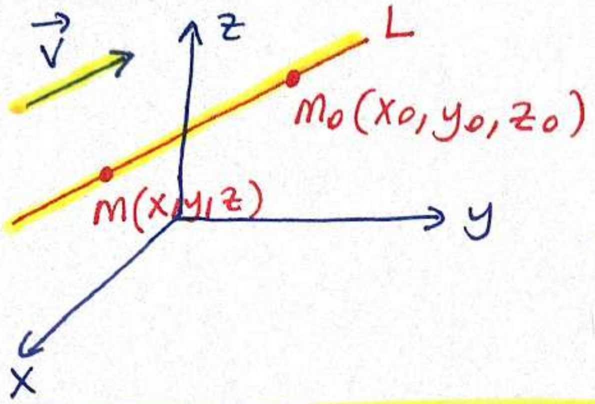
m is any nonzero constant

$\implies \boxed{\begin{aligned} A_1 &= m\,A_2 \\ B_1 &= m\,B_2 \\ C_1 &= m\,C_2 \end{aligned}}$

In addition, if $\boxed{D_1 = m\,D_2}$,

the planes are coincident.

In 3D :



$\overrightarrow{M_0M} \uparrow\uparrow \vec{v}$

$\vec{v} = \langle a,b,c \rangle$

$\overrightarrow{M_0M} = \langle x-x_0, y-y_0, z-z_0 \rangle$

$$L: \frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

$\vec{v}(0,b,c)$; $x=x_0$, $\frac{y-y_0}{b} = \frac{z-z_0}{c}$ a line equation orthogonal to axis x,

$\vec{v}(a,0,c)$; $\frac{x-x_0}{a} = \frac{z-z_0}{c}$, $y=y_0$ orthogonal to axis y,

$\vec{v}(a,b,0)$; $\frac{x-x_0}{a} = \frac{y-y_0}{b}$, $z=z_0$ orthogonal to axis z,

$\vec{v}(0,0,c)$; $x=x_0, y=y_0$, $\frac{z-z_0}{c}$ orthogonal to axes x and y, parallel to axis z,

$\vec{v}(0,b,0)$; $x=x_0$, $\frac{y-y_0}{b}$, $z=z_0$ orthogonal to axes x and z, parallel to axis y,

$\vec{v}(a,0,0)$; $\frac{x-x_0}{a}$, $y=y_0$, $z=z_0$ orthogonal to axes y and z, parallel to axis x.

Ex. Find the line equation which passing through point $P(1,2,3)$, and parallel to vector $\overrightarrow{AB} = \langle 3,1,-2 \rangle$.

$$L: \frac{x-1}{3} = \frac{y-2}{1} = \frac{z-3}{-2}.$$

Ex. Find the plane equation passing through $P_0(-2,-1,-3)$ and orthogonal to the line $\frac{x-1}{2}=\frac{y+3}{5}=\frac{z-2}{3}$.

$\vec{v}=\langle 2,5,3\rangle$

$Ax+By+Cz+D=0$

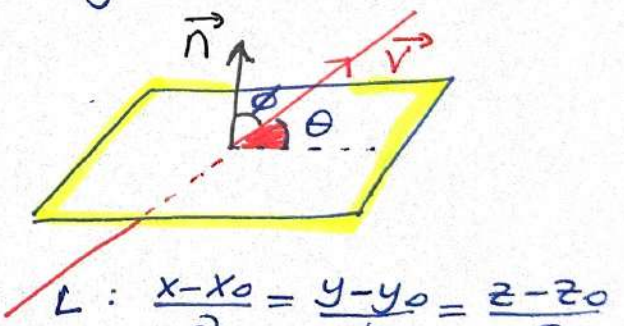
$2x_0+5y_0+3z_0+D=0$, $P_0(x_0,y_0,z_0)$

$2(-2)+5(-1)+3(-3)+D=0$

$-4-5-9+D=0$

$D=18 \Rightarrow 2x+5y+3z+18=0.$

Ex. Find the angle θ between the vector $\vec{v}=\langle a,b,c\rangle$ on a line L and a plane $Ax+By+Cz+D=0$.



$L: \frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}$

$\theta+\phi=90^\circ \Rightarrow \theta=90^\circ-\phi$

$\vec{v}\cdot\vec{n}=aA+bB+cC$

$\|\vec{v}\|=\sqrt{a^2+b^2+c^2}$

$\|\vec{n}\|=\sqrt{A^2+B^2+C^2}$

$\vec{v}\cdot\vec{n}=\|\vec{v}\|.\|\vec{n}\|.\cos\phi$ ($\vec{v},\vec{n}$ non-zero vectors)

$$\cos\phi=\frac{\vec{v}\cdot\vec{n}}{\|\vec{v}\|.\|\vec{n}\|}$$

$$\cos\phi=\frac{aA+bB+cC}{\sqrt{a^2+b^2+c^2}\sqrt{A^2+B^2+C^2}}$$

$$\phi=\arccos\left(\frac{aA+bB+cC}{\sqrt{a^2+b^2+c^2}\sqrt{A^2+B^2+C^2}}\right)$$

$\theta=90^\circ-\phi.$

Ex. Find an explicit relation between x and y eliminating the parameter t. Plot the parametric equation

$$x(t) = 4\cos t + 2\ ,\ y(t) = 5\sin t - 1\ ,\quad 0 \le t \le \pi\ .$$

Hint: $\cos^2\alpha + \sin^2\alpha = 1$.

$$\left.\begin{array}{l} x = 4\cos t + 2 \implies \dfrac{x-2}{4} = \cos t \\ y = 5\sin t - 1 \implies \dfrac{y+1}{5} = \sin t \end{array}\right\}$$ taking square, then adding two eqs.

$$\implies \left(\frac{x-2}{4}\right)^2 + \left(\frac{y+1}{5}\right)^2 = \cos^2 t + \sin^2 t$$

$$\implies \left(\frac{x-2}{4}\right)^2 + \left(\frac{y+1}{5}\right)^2 = 1$$ (ellipse in $\mathbb{R}^2$) $C(2,-1)$

$a = 4 \to$ minor axis along the x axis, $b = 5 \to$ major axis along the y axis, Since $a < b$!

$$\boxed{\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1}$$ general ellipse equation in $\mathbb{R}^2$

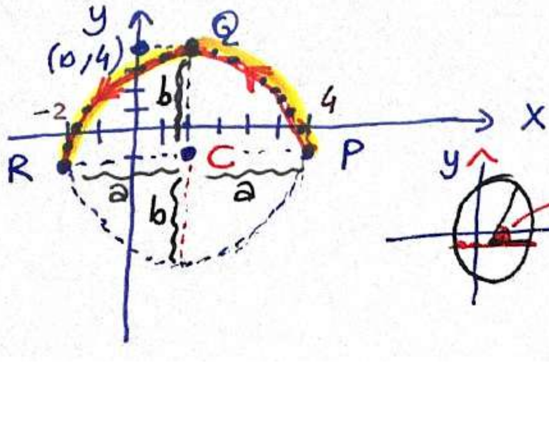
$C(h,k) \to$ center point

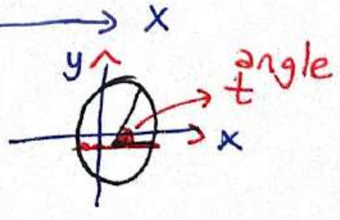
$t = 0 \implies x(0) = 4\cdot\cos(0) + 2 = 4(1) + 2 = 6$, $y(0) = 5\sin(0) - 1 = 5(0) - 1 = -1$ $\}$ $P(6,-1)$

$t = \frac{\pi}{2} \implies x(\frac{\pi}{2}) = 4\cos(\frac{\pi}{2}) + 2 = 4(0) + 2 = 2$, $y(\frac{\pi}{2}) = 5\cdot\sin(\frac{\pi}{2}) - 1 = 5(1) - 1 = 4$ $\}$ $Q(2,4)$

$t = \pi \implies x(\pi) = 4\cos(\pi) + 2 = 4\cdot(-1) + 2 = -2$, $y(\pi) = 5\sin(\pi) - 1 = 5\cdot(0) - 1 = -1$ $\}$ $R(-2,-1)$

Upper ellipse (red one) from point P to R.





Ex. Find the equation for the line passing through $(3,2,5)$ and parallel to the intersecting line between the planes

$$0.x + 3y - 2z = 7 \quad \text{and} \quad x - 4y \overset{+0.z}{=} 6.$$

Normal vector of the first plane : $\vec{n_1} = \langle 0,3,-2 \rangle$,

normal vector of the second plane : $\vec{n_2} = \langle 1,-4,0 \rangle$.

The direction vector of the line of intersection is given by the cross product of $\vec{n_1}$ and $\vec{n_2}$:

$$\vec{v} = \vec{n_1} \times \vec{n_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 3 & -2 \\ 1 & -4 & 0 \end{vmatrix}$$

$$\vec{v} = [3.0 - (-2)(-4)]\vec{i} - [0.0 - 1.(-2)]\vec{j} + [0.(-4) - 1.3]\vec{k}$$

$$\vec{v} = (0-8)\vec{i} - (0+2)\vec{j} + (0-3)\vec{k} = -8\vec{i} - 2\vec{j} - 3\vec{k}$$

$$\vec{v} = \langle -8,-2,-3 \rangle.$$

The line passing through $(3,2,5)$ and parallel to $\vec{v} = \langle -8;-2,-3 \rangle$ is determined by direction vector

$$x(t) = 3 + 8t$$
$$y(t) = 2 + 2t \qquad \text{(parametric eq.)}$$
$$z(t) = 5 + 3t$$

or

$$\frac{x-3}{8} = \frac{y-2}{2} = \frac{z-5}{3} \qquad \text{(symmetric eq.)}.$$

{ Here, we take $-\vec{v}$ not $\vec{v}$. But you can also use $\vec{v}$. }

Ex. Find the equation for the plane passing through points $P(2,4,-1)$, $Q(-3,2,6)$, and $O(0,0,0)$. Hint: Find the normal vector $\vec{N}$ first.

$$\vec{PQ} = \langle -3-2,\, 2-4,\, 6-(-1)\rangle = \langle -5,-2,7\rangle,$$

$$\vec{PO} = \langle 0-2,\, 0-4,\, 0-(-1)\rangle = \langle -2,-4,1\rangle.$$

Normal vector $\vec{N}$ to the plane is given by

$$-\vec{N} = \vec{PQ} \times \vec{PO} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -5 & -2 & 7 \\ -2 & -4 & 1 \end{vmatrix}$$

$$-\vec{N} = [(-2).1-(-4).7]\,\vec{i} - [(-5).1-(-2)7]\,\vec{j} + [(-5)(-4)-(-2)(-2)]\,\vec{k}$$

$$-\vec{N} = (-2+28)\,\vec{i} - (-5+14)\,\vec{j} + (20-4)\,\vec{k}$$

$$-\vec{N} = 26\,\vec{i} - 9\,\vec{j} + 16\,\vec{k}$$

$$\vec{N} = \langle -26, +9, -16\rangle$$

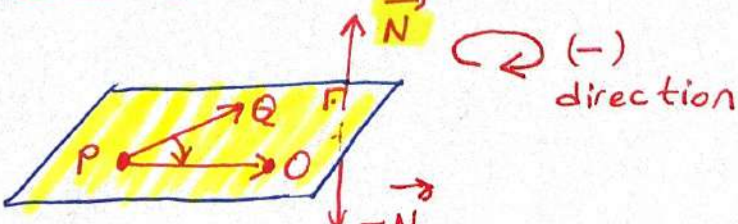
★ The general equation of a plane with normal vector $\vec{N} = \langle A,B,C\rangle$ passing through a point $P_0(x_0,y_0,z_0)$ is described by $A(x-x_0)+B(y-y_0)+C(z-z_0)=0$.

Using point $P(2,4,-1)$ and normal vector $\vec{N} = \langle 26,-9,16\rangle$

$$-26(x-2)+9(y-4)-16(z+1)=0$$

$$\Rightarrow -26x+52+9y-36-16z-16=0$$

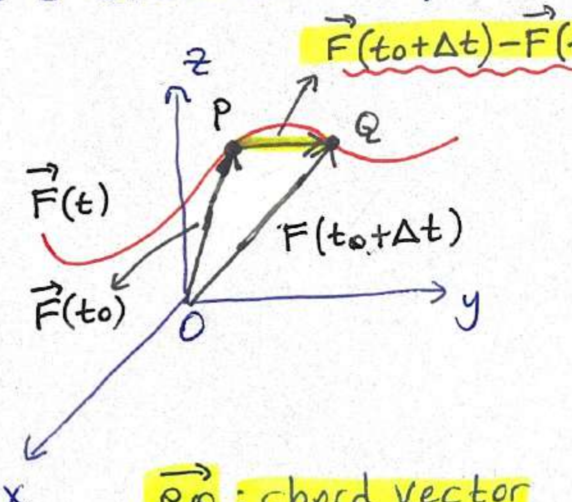
$$\Rightarrow -26x+9y-16z=0. \quad \text{(the eq. of the plane)}$$



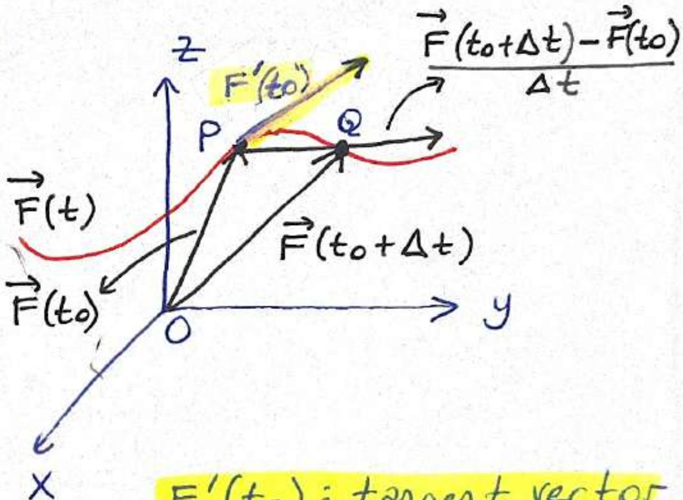
Tangent Vectors :

The derivative $f'(x_0)$ gives the slope of the tangent line to the graph at x_0.

Similarly, for a vector function $\vec{F}(t)$, the derivative $\vec{F}'(t_0)$ provides a tangent vector to the curve at the point corresponding to t_0, indicating the curve's direction in space.



$\vec{PQ}$: chord vector



$F'(t_0)$: tangent vector

☆ Let $\vec{F}(t)$ be differentiable at t_0 with $\vec{F}'(t_0) \neq 0$. In this case, $\vec{F}'(t_0)$ represents a tangent vector to the graph of $\vec{F}(t)$ at $t = t_0$, pointing in the direction of increasing t.

$$F'(t_0) = \lim_{\Delta t \to 0} \frac{\Delta \vec{F}}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\vec{F}(t_0+\Delta t) - \vec{F}(t_0)}{\Delta t}$$

Ex. Find a tangent vector at point P, where $t=0.1$ for $\vec{F}(t) = \langle e^{3t}, t^3-t, \ln t \rangle$.
What is an equation for the line at P?

$$\vec{F}'(t) = \langle 3 \cdot e^{3t}, 3t^2-1, \frac{1}{t} \rangle$$
$$\vec{F}'(0.1) = \langle 3 \cdot e^{3 \cdot (0.1)}, 3(0.1)^2-1, \frac{1}{0.1} \rangle$$
$$\vec{F}'(0.1) = \langle 3e^{0.3}, -0.97, 10 \rangle$$

$$\vec{OP} = \vec{F}(0.1) = \langle e^{3(0.1)}, (0.1)^3-(0.1), \ln(0.1) \rangle$$
$$\vec{OP} = \langle e^{0.3}, -0.099, -2.303 \rangle.$$

Then, the equation of the tangent line is
$$\vec{r}(t) = \vec{OP} + t \cdot \vec{F}'(0.1)$$
$$\Rightarrow \quad \vec{r}(t) = \langle e^{0.3} + 3te^{0.3}, -0.099 - 0.97t, -2.303 + 10t \rangle.$$

If $F'(t_0) \neq 0$ and is continuous at t_0, the tangent vectors near P will be close to the tangent vector at P. This means the graph is smooth at P.

A smooth curve is the graph of a vector function $\vec{F}(t)$ on any interval where $\vec{F}'(t) \neq 0$ is continuous.
The graph is considered piecewise smooth if the interval can be divided into a finite number of subintervals where $\vec{F}$ is smooth.

Ex. Describe whether the graph of a function

$$\vec{F}(t) = (t^3+2)\vec{i} + \sin t\,\vec{j} + (e^t+e^{-t})\vec{k}$$

is smooth for all t.

1. • The derivative $F'(t)$ must be continuous,
2. • $\vec{F}'(t) \neq 0$ for all t.

$$\left.\begin{aligned} &\frac{d}{dt}(t^3+2) = 3t^2, \\ &\frac{d}{dt}(\sin t) = \cos t, \\ &\frac{d}{dt}(e^t-e^{-t}) = e^t-(-1)e^{-t} = e^t+e^{-t}. \end{aligned}\right\} \text{Thus,}$$

$$\vec{F}'(t) = \langle 3t^2, \cos t, e^t+e^{-t} \rangle$$

All polynomial, trigonometric, exponential components of $\vec{F}'(t)$ are continuous for all t.

$\vec{F}'(0) = \langle 0, 1, 0 \rangle \neq \vec{0}$ for $t=0$.

For all t, $F'(t) \neq 0$. Then, $\vec{F}$ is smooth.

$$F'(t) = \frac{d\vec{F}(t)}{dt}, \quad F''(t) = \frac{d}{dt}\left(\frac{d\vec{F}(t)}{dt}\right) = \frac{d^2\vec{F}(t)}{dt^2},$$

$$F'''(t) = \frac{d}{dt}\left(\frac{d^2\vec{F}(t)}{dt^2}\right) = \frac{d^3\vec{F}(t)}{dt^3}, \ \ldots \ \left(\text{higher derivatives}\right)$$

Ex. $\vec{F}(t) = \langle t^3, \cos(3t), e^{4t} \rangle \Rightarrow F'''(t) = ?$

$$F'(t) = \langle 3t^2, -3\sin(3t), 4e^{4t} \rangle,$$

$$F''(t) = \langle 6t, -9\cos(3t), 16e^{4t} \rangle,$$

$$F'''(t) = \langle 6, 27\sin(3t), 64e^{4t} \rangle.$$

Rules for Differentiating :

Let $\vec{F}$ and $\vec{G}$ be vector functions, h be scalar function, and are differentiable functions on t. Then, $a\vec{F}+b\vec{G}$, $h\vec{F}$, $\vec{F}\cdot\vec{G}$, $\vec{F}\times\vec{G}$ are also differentiable at t;

$(a\vec{F}+b\vec{G})'(t) = a\vec{F}'(t) + b\vec{G}'(t)$ for $a,b \in \mathbb{R}$, (linearity)

$(h.\vec{F})'(t) = h'(t)\vec{F}(t) + h(t)\vec{F}'(t)$ (scalar multiple)

$(\vec{F}\cdot\vec{G})'(t) = (\vec{F}'\cdot\vec{G})(t) + (\vec{F}\cdot\vec{G}')(t)$ (dot product)

$(\vec{F}\times\vec{G})'(t) = (\vec{F}'\times\vec{G})(t) + (\vec{F}\times\vec{G}')(t)$ (cross product)

$[\vec{F}(h(t))]' = h'(t).\vec{F}'(h(t))$ (chain)

If the non-zero vector function $\vec{F}(t)$ is differentiable and has a constant length, then $\vec{F}(t) \perp \vec{F}'(t)$.

$\|\vec{F}(t)\| = c$ (for some constant c, all t)

$\vec{F}(t)\cdot\vec{F}'(t) = 0$ (?)

$$\|\vec{F}(t)\|^2 = c^2 = \vec{F}(t)\cdot\vec{F}(t)$$

$$\Rightarrow (c^2)' = [\vec{F}(t)\cdot\vec{F}(t)]'$$

$$\Rightarrow 0 = \vec{F}'(t)\cdot\vec{F}(t) + \vec{F}(t)\cdot\vec{F}'(t)$$

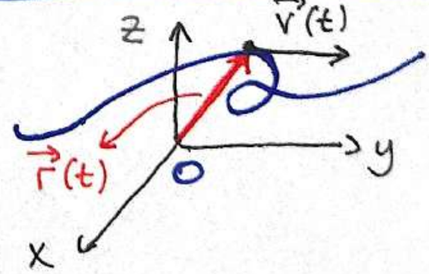
$$\Rightarrow 0 = 2(\vec{F}(t)\cdot\vec{F}'(t))$$

$$\Rightarrow \vec{F}(t)\cdot\vec{F}'(t) = 0$$

$$\Rightarrow \vec{F}(t) \perp \vec{F}'(t).$$

Vector Motion:

- An object's position at time t is given by the vector function $\vec{r}(t)$,
- The position vector is $\vec{r}(t)$, and the velocity is $\vec{v} = \frac{d\vec{r}}{dt}$,
- The velocity vector is tangent to the object's trajectory,
- At any time t, the speed is $\|\vec{v}\|$, the magnitude of the velocity,
- The direction of motion is given by the unit vector: $\frac{\vec{v}}{\|\vec{v}\|}$,
- The acceleration is the derivative of velocity $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$,
- At point P_0, the tangent vector $\vec{r}\,'(t_0)$ is orthogonal to the radius vector $\vec{r}(t_0)$



$$\left.\begin{array}{ll} \vec{r}(t) & : \text{Position vector} \\ \vec{v} = \frac{d\vec{r}}{dt} & : \text{velocity} \\ \|\vec{v}\| & : \text{speed} \\ \frac{\vec{v}}{\|\vec{v}\|} & : \text{direction of motion} \\ \vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt} & : \text{acceleration vector} \end{array}\right\} \text{at any time } t.$$

Position vector :

$$\vec{r}(t) = \langle f_1(t), f_2(t), f_3(t) \rangle ,$$

Velocity vector :

$$\vec{v}(t) = \vec{r}\,'(t) = \langle f_1'(t), f_2'(t), f_3'(t) \rangle ,$$

Speed :

$$\| \vec{v}(t) \| = \| r'(t) \| = \sqrt{[f_1'(t)]^2 + [f_2'(t)]^2 + [f_3'(t)]^2} ,$$

Acceleration vector :

$$\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t) = \langle f_1''(t), f_2''(t), f_3''(t) \rangle .$$

Direction of motion (is the unit vector) :

$$\frac{\vec{v}}{\|\vec{v}\|} .$$

Ex. Using a particle's position $\vec{r}(t) = \langle t^2, \sin t, \cos t \rangle$, find the velocity, speed, acceleration and direction of motion at $t=1$.

$$\vec{v}(t) = \vec{r}\,'(t) = \langle 2t, \cos t, -\sin t \rangle$$

$$\|\vec{v}(t)\| = \sqrt{(2t)^2 + (\cos t)^2 + (\sin t)^2} = \sqrt{4t^2+1}$$

$$\|\vec{v}(1)\| = \sqrt{4 \cdot (1)^2 + 1} = \sqrt{5} \quad (t=1), \qquad \sqrt{5} \approx 2.24$$

$$\vec{a}(t) = \vec{r}\,''(t) = \langle 2, -\sin t, -\cos t \rangle ,$$

at $t=1$:

$$\vec{v}(1) = \langle 2 \cdot (1), \cos(1), -\sin(1) \rangle = \langle 2, 0.54, 0.84 \rangle$$

$$\frac{\vec{v}(1)}{\|\vec{v}(1)\|} = \frac{\langle 2, 0.54, 0.84 \rangle}{\sqrt{5}} \doteq \left\langle \frac{2}{2.24}, \frac{0.54}{2.24}, \frac{0.84}{2.24} \right\rangle .$$

Vector Integrals: Let $\vec{F}(t) = \langle f_1(t), f_2(t), f_3(t) \rangle$,

where $f_1(t), f_2(t), f_3(t)$ are continuous on $\alpha \le t \le \beta$.

The **indefinite integral**:

$$\boxed{\int \vec{F}(t)\,dt = \left\langle \int f_1(t)\,dt,\ \int f_2(t)\,dt,\ \int f_3(t)\,dt \right\rangle + \vec{C}.}$$

Here, $\vec{C} = \langle c_1, c_2, c_3 \rangle$ any constant vector.

Similarly, **definite integral**:

$$\boxed{\int_\alpha^\beta \vec{F}(t)\,dt = \left\langle \int_\alpha^\beta f_1(t)\,dt,\ \int_\alpha^\beta f_2(t)\,dt,\ \int_\alpha^\beta f_3(t)\,dt \right\rangle.}$$

Ex. Find the integral, using $\vec{F}(t) = \underbrace{t^2}_{f_1(t)}\vec{i} - \underbrace{3e^t}_{f_2(t)}\vec{j} + \underbrace{\cos t}_{f_3(t)}\vec{k}$:

$$\int_0^{2\pi} \vec{F}(t)\,dt.$$

$$\int_0^{2\pi} f_1(t) = \int_0^{2\pi} (t^2)\,dt = \frac{t^3}{3}\Big|_0^{2\pi} = \frac{(2\pi)^3}{3} = \frac{8\pi^3}{3},$$

$$\int_0^{2\pi} f_2(t) = \int_0^{2\pi} (-3e^t)\,dt = -3e^t\Big|_0^{2\pi} = -3\left(e^{2\pi} - e^0\right) = -3\left(e^{2\pi} - 1\right),$$

$$\int_0^{2\pi} f_3(t) = \int_0^{2\pi} (\cos t)\,dt = -\sin t\Big|_0^{2\pi} = -(\underbrace{\sin 2\pi}_{=0} - \underbrace{\sin 0}_{=0}) = 0.$$

Then, $\displaystyle\int_0^{2\pi} \vec{F}(t)\,dt = \frac{8}{3}\pi^3\,\vec{i} - 3\left(e^{2\pi} - 1\right)\vec{j}.$

★ Ex. Let the velocity vector be

$$\vec{v}(t) = \cos t\,\vec{i} + e^{3t}\,\vec{j} - 3t\,\vec{k}$$

and the initial position vector be

$$\vec{r}(0) = 2\vec{i} - 3\vec{j} + \vec{k}\,.$$

Compute the acceleration vector $\vec{a}(t)$, and the position vector $\vec{r}(t)$.

$$\vec{a}(t) = \vec{v}'(t) = \langle -\sin t,\ 3e^{3t},\ -3 \rangle$$

$$\vec{r}'(t) = \vec{v}(t) \implies \vec{r}(t) = \int \vec{v}(t)\,dt + \vec{C}$$

$$\implies \vec{r}(t) = \int (\cos t\,\vec{i} + e^{3t}\vec{j} - 3t\,\vec{k})\,dt + \vec{C}$$

$$\implies \vec{r}(t) = \langle \sin t,\ \tfrac{1}{3}e^{3t},\ -\tfrac{3t^2}{2} \rangle + \langle c_1, c_2, c_3 \rangle$$

or $\vec{r}(t) = (\sin t + c_1)\vec{i} + \left(\frac{1}{3}e^{3t} + c_2\right)\vec{j} + \left(-\frac{3t^2}{2} + c_3\right)\vec{k}$,

at $t=0$: $\vec{r}(0) = (\sin(0) + c_1)\vec{i} + \left(\frac{1}{3}e^{3(0)} + c_2\right)\vec{j} + \left(-\frac{3}{2}(0)^2 + c_3\right)\vec{k}$.

We know: $\vec{r}(0) = 2\vec{i} - 3\vec{j} + 1\vec{k}$. Then,

$$\left.\begin{aligned} 0 + c_1 &= 2 \\ \tfrac{1}{3} + c_2 &= -3 \\ 0 + c_3 &= 1 \end{aligned}\right\} \qquad \begin{aligned} c_1 &= 2 \\ c_2 &= -3 - \tfrac{1}{3} = -\tfrac{10}{3} \\ c_3 &= 1\,. \end{aligned}$$

- $\vec{r}(t) = (\sin t + 2)\vec{i} + \left(\frac{1}{3}e^{3t} - \frac{10}{3}\right)\vec{j} + \left(-\frac{3}{2}t^2 + 1\right)\vec{k}$,

and

- $\vec{a}(t) = (-\sin t)\vec{i} + 3e^{3t}\vec{j} + (-3)\vec{k}$.

Unit Tangent Vector and Unit Normal Vector :

For a smooth curve defined by $\vec{r}(t)$:

- The unit tangent vector is $\boxed{\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}}$,
- The principal unit normal vector is $\boxed{\vec{N}(t) = \dfrac{\vec{T}\,'(t)}{\|\vec{T}\,'(t)\|}}$.

Ex. Determine the unit tangent vector $\vec{T}(t)$ and the principal unit normal vector $\vec{N}(t)$ for the vector function $\vec{r}(t) = \langle 2\cos t, t^2, 2\sin t \rangle$ at each point on its curve.

$$\vec{r}\,'(t) = \langle -2\sin t, 2t, 2\cos t \rangle$$

$$\|\vec{r}\,'(t)\| = \sqrt{4\sin^2 t + 4t^2 + 4\cos^2 t}$$

$$= \sqrt{4(\sin^2 t + \cos^2 t) + 4t^2}$$

$$= \sqrt{4 + 4t^2} = \sqrt{4(1+t^2)} = 2\sqrt{1+t^2}$$

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|r'(t)\|} = \frac{\langle -2\sin t, 2t, 2\cos t \rangle}{2\sqrt{1+t^2}}$$

$$\Rightarrow \vec{T}(t) = \left\langle -\frac{\sin t}{\sqrt{1+t^2}}, \frac{t}{\sqrt{1+t^2}}, \frac{\cos t}{\sqrt{1+t^2}} \right\rangle.$$

$$\frac{d}{dt}\left(\frac{-\sin t}{\sqrt{1+t^2}}\right) = \frac{-\cos t\left(\sqrt{1+t^2}\right) - (-\sin t)\frac{t}{\sqrt{1+t^2}}}{1+t^2} = \frac{(-\cos t)(1+t^2) + t\sin t}{(1+t^2)^{3/2}},$$

$$\frac{d}{dt}\left(\frac{t}{\sqrt{1+t^2}}\right) = \frac{1.\sqrt{1+t^2} - t.\frac{t}{\sqrt{1+t^2}}}{1+t^2} = \frac{1}{(1+t^2)^{3/2}},$$

$$\frac{d}{dt}\left(\frac{\cos t}{\sqrt{1+t^2}}\right) = \frac{(-\sin t)\sqrt{1+t^2} - \cos t\,\frac{t}{\sqrt{1+t^2}}}{1+t^2} = \frac{(-\sin t)(1+t^2) - t\cos t}{(1+t^2)^{3/2}}$$

$$\|\vec{T}'(t)\| = \sqrt{\left[\frac{(-\cos t)(1+t^2)+t\sin t}{(1+t^2)^{3/2}}\right]^2+\left[\frac{1}{(1+t^2)^{3/2}}\right]^2+\left[\frac{(-\sin t)(1+t^2)-t\cos t}{(1+t^2)^{3/2}}\right]^2}$$

$$\Rightarrow \quad \vec{N}(t) = \frac{\vec{T}'(t)}{\|T'(t)\|}.$$

Parametric Arc Length

In $\mathbb{R}^2$ (or in 2D), arclength of any curve is given by (for a curve $y=f(x)$) on interval $\alpha \leq x \leq \beta$:

$$s = \int_{\alpha}^{\beta} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx .$$

When $x=x(t)$, $y=y(t)$ (parametric form),

$$\frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt} \quad \text{then,} \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \quad \left(\frac{dx}{dt} \neq 0\right)$$

So, for $t_1 \leq t \leq t_2$, arclength is determined by

$$(\alpha \leq x \leq \beta) \quad s = \int_{\alpha}^{\beta} \sqrt{1+\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)^2}\, dx = \int_{t_1}^{t_2} \sqrt{\frac{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}{\left(\frac{dx}{dt}\right)^2}}\cdot\left(\frac{dx}{dt}\right)dt$$

$$\Rightarrow \boxed{s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\, dt.} \quad \text{(in 2D)}$$

Then, in $\mathbb{R}^3$ parametric arclength is defined by

$$\star \quad \boxed{s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\, dt} \quad \text{(in 3D)}$$

for $x=x(t)$, $y=y(t)$, $z=z(t)$, and $t_1 \leq t \leq t_2$.

Ex. Find the arclength of the curve

$$\vec{R}(t) = \langle \underbrace{4e^{-3t}\cdot\cos t}_{x(t)}, \underbrace{4\cdot e^{-3t}\sin t}_{y(t)}, \underbrace{2e^{-3t}}_{z(t)} \rangle$$

for $0 \le t \le 2$. <u>Hint</u>: Simplify, simplify, then integrate.

$$s = \int_0^2 \sqrt{\Big[\underbrace{\tfrac{d}{dt}\left(4e^{-3t}\cos t\right)}_{x'(t)}\Big]^2 + \Big[\underbrace{\tfrac{d}{dt}\left(4e^{-3t}\sin t\right)}_{y'(t)}\Big]^2 + \Big[\underbrace{\tfrac{d}{dt}\left(2e^{-3t}\right)}_{z'(t)}\Big]^2}\,dt$$

$$\frac{d}{dt}\left[4e^{-3t}\cos t\right] = -12e^{-3t}\cos t + 4e^{-3t}(-\sin t)$$

$$\frac{d}{dt}\left[4e^{-3t}\sin t\right] = -12e^{-3t}\sin t + 4e^{-3t}(\cos t)$$

$$\frac{d}{dt}\left[2e^{-3t}\right] = -6e^{-3t}$$

$$\left\{ \begin{aligned} x'(t) &= \frac{d}{dt}x(t) \\ y'(t) &= \frac{d}{dt}y(t) \\ z'(t) &= \frac{d}{dt}z(t) \end{aligned} \right\}$$

$$\Rightarrow \vec{R}'(t) = \langle 4e^{-3t}(-3\cos t - \sin t), 4e^{-3t}(-3\sin t + \cos t), -6e^{-3t} \rangle$$

$$\|\vec{R}'(t)\| = \sqrt{16e^{-6t}(-3\cos t - \sin t)^2 + 16e^{-6t}(-3\sin t + \cos t)^2 + 36e^{-6t}}$$

$$= e^{-3t}\sqrt{16(9\cos^2 t + \cancel{3\sin t\cos t} + \sin^2 t) + 16(9\sin^2 t - \cancel{3\sin t\cos t} + \cos^2 t) + 36}$$

$$= e^{-3t}\sqrt{16\Big[9.(\underbrace{\sin^2 t + \cos^2 t}_{=1}) + (\underbrace{\sin^2 t + \cos^2 t}_{=1})\Big] + 36}$$

$$= e^{-3t}\sqrt{16(9+1)+36} = e^{-3t}\sqrt{196}$$

$$\Rightarrow \|\vec{R}'(t)\| = 14.e^{-3t}$$

$$s = \int_0^2 14.e^{-3t}\,dt = 14.\left(-\frac{1}{3}\right)e^{-3t}\Big|_0^2$$

$$s = -\frac{14}{3}\left(e^{-3.(2)} - e^{-3.(0)}\right) = \underline{-\frac{14}{3}\left(e^{-6} - 1\right)}$$

Ex. Reparametrize the helix

$$\vec{r}(t) = \langle \cos t, \sin t, t \rangle$$

using the arclength s, with s measured from the point $P_0 = (1,0,0)$ as t increases.

$$r'(t) = \langle -\sin t, \cos t, 1 \rangle$$

$$\|\vec{r}'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1} = \sqrt{\underbrace{\sin^2 t + \cos^2 t}_{=1} + 1} = \sqrt{2}$$

$$s(t) = \int_0^t \|\vec{r}'(\alpha)\| \, d\alpha$$

$$s(t) = \int_0^t \sqrt{2} \, d\alpha = \sqrt{2}\, t \implies s = \sqrt{2}\, t \implies t = \frac{s}{\sqrt{2}}$$

The point $P_0 = (1,0,0)$ corresponds to $t=0$ in the original parametrization because

$$\vec{r}(0) = \langle \cos(0), \sin(0), 0 \rangle = \langle 1,0,0 \rangle.$$

Then,

$$\vec{r}(s) = \left\langle \cos\left(\frac{s}{\sqrt{2}}\right), \sin\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}} \right\rangle.$$

For a piecewise-smooth curve $\vec{r}(s)$ parametrized by arc length s:

- The unit tangent vector is $\boxed{\vec{T} = \frac{d\vec{r}}{ds}}$,
- The principal unit normal vector satisfies

$$\boxed{\frac{d\vec{T}}{ds} = \kappa \vec{N}},$$ where κ is the curvature.

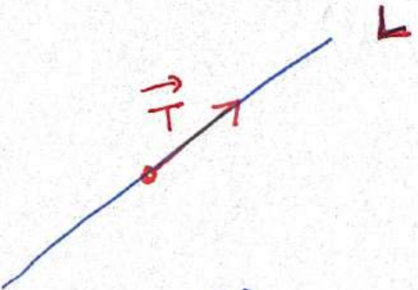
Here, $\boxed{\kappa = \left\| \frac{d\vec{T}}{ds} \right\|}$. Here, $\kappa = \kappa(s)$.

An object moves along a smooth curve C, with position function $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ and continuous $\vec{r}\,'(t)$ on $[t_1, t_2]$. Its speed is

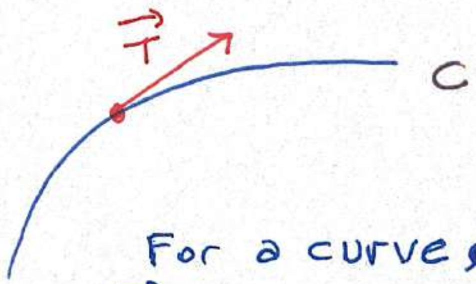
$$\frac{ds}{dt} = \|\vec{v}(t)\| = \|\vec{r}\,'(t)\| \qquad \left(s = \int \|\vec{v}(t)\| dt = \int \|\vec{r}\,'(t)\| dt \right)$$

A smooth curve C, parametrized by arc length s, has a unit tangent vector $\vec{T}(s)$ whose direction changes with s. The rate of this change, $\frac{d\vec{T}}{ds}$, measures the curvature of the curve.

A straight line has zero curvature, while sharper bends results in greater curvature.



For a line L, $\frac{d\vec{T}}{ds} = 0$



For a curve C, $\left\| \frac{d\vec{T}}{ds} \right\| = \kappa$ is large where the curve is sharply bent.

$$\frac{ds}{dt} = \|\vec{r}\,'(t)\| \Rightarrow \kappa = \frac{\| T'(t) \|}{\| \vec{r}\,'(t) \|}.$$

Ex. Find the curvature of the circle with radius a.

$\vec{r}(t) = \langle a\cos t, a\sin t \rangle$

$\vec{r}'(t) = \langle -a\sin t, a\cos t \rangle$

$\|\vec{r}'(t)\| = \sqrt{(-a\sin t)^2 + (a\cos t)^2} = a$

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|r'(t)\|} = \frac{\langle -a\sin t, a\cos t \rangle}{a} = \langle -\sin t, \cos t \rangle$$

$T'(t) = \langle -\cos t, -\sin t \rangle$, $\|T'(t)\| = 1$

$$\kappa = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|} = \frac{1}{a}.$$

Let C be a smooth curve and graph of $\vec{r}(t)$.

Then,
$$\boxed{\kappa = \frac{\vec{r}' \times \vec{r}''}{\|r'\|^3}} \quad \left\{ \text{curvature formula} \right\} \quad \left\{ \kappa = \kappa(t) \right\}$$

Ex. Find the curvature of the helix
$$\vec{r}(t) = \langle a\cos t, a\sin t, bt \rangle.$$

Hint. Use the above formula of κ.

The graph C of $y = f(x)$ has
$$\boxed{\kappa = \frac{|y''|}{\left[1+(y')^2\right]^{3/2}}}, \quad \left\{ \kappa = \kappa(x) \right\}$$

when f, f', f'' (or y, y', y'') all exist.

Ex. Find the x coordinate of the point of maximum curvature (call it x_0) on the curve $y = a \cdot e^{bx}$ (for any constants a and b) and find the maximum curvature, $\kappa(x_0)$.

$$\left.\begin{aligned} y &= f(x) = a \cdot e^{bx} \\ y' &= a \cdot b \cdot e^{bx} \\ y'' &= a \cdot b^2 \cdot e^{bx} \end{aligned}\right\}, \quad \kappa = \frac{|y''|}{\left[1+(y')^2\right]^{3/2}}, \quad \kappa = \kappa(x).$$

$$\kappa(x) = \frac{ab^2 \cdot e^{bx}}{(1+a^2b^2e^{2bx})^{3/2}}$$

$$\kappa'(x) = \frac{-ab^3e^{bx}\left(2a^2b^2e^{2bx}-1\right)}{(1+a^2b^2e^{2bx})^{5/2}} \quad \left\{ \begin{array}{c} \text{from quotient rule} \\ \left(\frac{f}{g}\right)' = \frac{f'g-g'f}{g^2} \end{array} \right\}$$

$$\kappa'(x) = 0 \Rightarrow 2a^2b^2e^{2bx_0} - 1 = 0$$

$$\Rightarrow e^{2bx_0} = \frac{1}{2a^2b^2}$$

$$\Rightarrow 2bx_0 = \ln\left(\frac{1}{2a^2b^2}\right)$$

$$\Rightarrow \boxed{x_0 = \frac{1}{2b}\ln\left(\frac{1}{2a^2b^2}\right)}. \quad \left\{ \begin{array}{c} x_0 \text{ gives the} \\ x \text{ coordinate} \\ \text{of the point} \\ \text{of maximum} \\ \text{curvature} \end{array} \right\}$$

Then, substituting x_0 into $\kappa(x)$:

$$\boxed{\kappa(x_0) = \frac{ab^2\, e^{b(x_0)}}{(1+a^2b^2e^{2b(x_0)})^{3/2}}} \quad \left\{ \begin{array}{c} \text{maximum} \\ \text{curvature} \end{array} \right\}$$

Curvature Formulas

	Formula	Given	type
1)	$\kappa(s) = \left\lVert \frac{d\vec{T}}{ds} \right\rVert$	$\vec{r}(s)$	s: arclength parameter
2)	$\kappa(t) = \left\lVert \frac{\vec{T}'(t)}{\vec{r}'(t)} \right\rVert$	$\vec{r}(t)$	two-derivatives
3)	$\kappa(t) = \frac{\lVert \vec{r}'(t) \times r''(t) \rVert}{\lVert \vec{r}'(t) \rVert^3}$	$r(t)$	cross-derivative
4)	$\kappa(x) = \frac{\lvert y'' \rvert}{[1+(y')^2]^{3/2}}$	$y=f(x)$	functional
5)	$\kappa(t) = \frac{\lvert x'y''-y'x'' \rvert}{[(x')^2+(y')^2]^{3/2}}$	$x=x(t)$, $y=y(t)$	parametric
6)	$\kappa(r) = \frac{\lvert r^2+2r'^2-rr'' \rvert}{(r^2+r'^2)^{3/2}}$	$r=f(\theta)$	polar

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\lVert r'(t) \rVert}, \quad N(t) = \frac{\vec{T}'(t)}{\lVert \vec{T}'(t) \rVert},$$

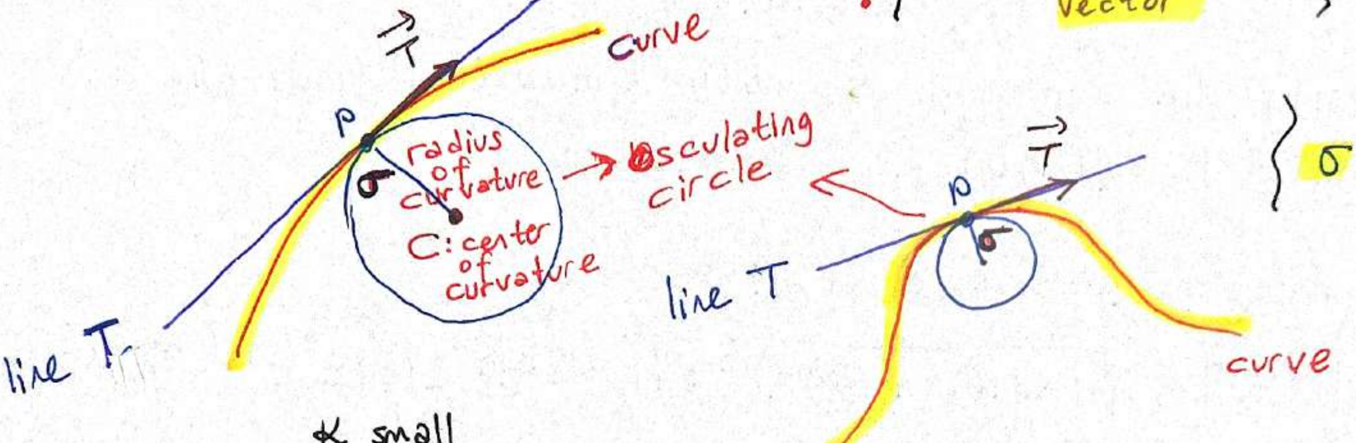
unit tangent vector

unit normal vector

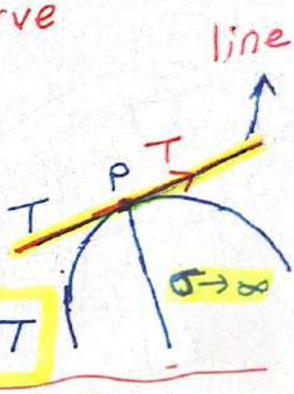
! T: tangent line

! $\vec{T}$: unit tangent vector

σ: radius (sigma)



κ small κ large



The curvature κ measures the rate at which the curve bends away from the line T at any point P.

When $\sigma \to \infty$, then $\kappa = 0$. So, curve reduces to a line.

Ex. Find the curvature κ of an helix

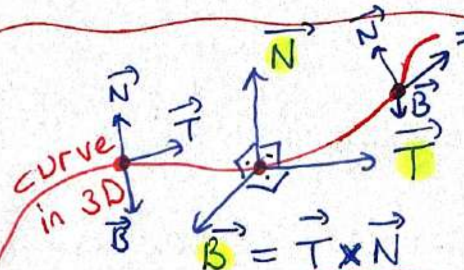
$$\vec{r}(t) = \langle a\cos t,\ a\sin t,\ bt \rangle$$

where $a, b \in \mathbb{R}^+$.

Hint: Find $\vec{r}\,'$, $\vec{r}\,''$ first. Then, $\vec{r}\,' \times \vec{r}\,''$ and its magnitude also magnitude of $\vec{r}\,'$.

Therefore, get $\kappa = \dfrac{\|\vec{r}\,' \times \vec{r}\,''\|}{\|\vec{r}\,'\|^3} = \dfrac{a}{a^2+b^2}$.

$\mathbb{R}^3$



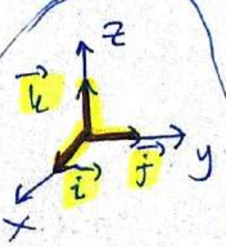
$\vec{B} = \vec{T} \times \vec{N}$

$\longleftrightarrow$ $\{\vec{T}, \vec{N}, \vec{B}\}$ → (moving) Frenet frame

orthogonal → just perpendicular vectors

also

orthonormal frame → perpendicular and unit vectors



$\{\vec{i}, \vec{j}, \vec{k}\}$ Euclidean standart frame (not-moving)